Twisted equiv. structs on bdles, Marwan BENYOUSSEF

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Γ discrete group, G Lie grp, $Z := Z(G)$

$\Gamma \curvearrowright G$ via $\theta : \Gamma \longrightarrow \mathrm{Aut}\, G$

$c \in Z^2_\theta(\Gamma, Z)$ a 2-cocycle for this action. ($c : \Gamma \times \Gamma \to Z$ verifying a cocycle condition)

$\hat{G} := G \times_{(\theta, c)} \Gamma$ (the extension of Γ by G given by θ, c)

is a group with underlying set is: $G \times \Gamma$ and operation:

$(g_1, \gamma_1) \bullet (g_2, \gamma_2) = (g_1 g_2^{\gamma_1} c(\gamma_1 \gamma_2), \gamma_1 \gamma_2)$, $\forall \gamma_1, \gamma_2 \in \Gamma$.

$\hat{\Gamma}_{\theta, c} := Z \times_{(\theta, c)} \Gamma$ (corresponding central extension of Γ by Z)

- we adopt the right action throughout.
- we omit the "$\cdot$" for the $G \curvearrowright E$, $E \longrightarrow X$ a principal G-bdle.

Def: A (θ, c)-twisted (G, Γ)-manifold is a smooth manif M with a (θ, c)-twisted (G, Γ)-action s.t. the maps defined by each $g \in G$ and $\gamma \in \Gamma$ are diffeo of M.

Def: Let X be a Γ-manifold.

A (θ, c)-twisted Γ-equiv. bundle on X is

- $\pi : F \longrightarrow X$ a fiber bundle.
- a (θ, c)-twisted action of Γ on F such that Z acts fiberwise and $\pi(f \cdot \gamma) = \pi(f) \cdot \gamma$, $\forall \gamma \in \Gamma$, $\forall f \in F$.

with obvious modif. in case of left Γ-action.

Let X be a right Γ-manifold.

Def: The category of (θ, c)-twisted Γ-equiv. principal G-bundles on X is defined by:

Objects:

A (θ, c)-twisted Γ-equiv. principal G-bdle on X is a right (θ, c)-twisted Γ-equiv. bdl $E \to X$ with a right G-action s.t.

1) The actions of G and Γ make E into a (θ, c)-twisted (G, Γ)-manifold

2) The action of G makes $E \to X$ into a principal G-bundle.

Morphisms: commutative diagrams:

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d] & F \arrow[d] \\ X \arrow[r, "f"] & X \end{tikzcd}$$

where

- f is Γ-equiv.,
- $\tilde{f}$ is (θ, c)-twisted (G, Γ)-equivariant

(i.e. $\tilde{f}(e.\gamma) = \tilde{f}(e).\gamma$, $\tilde{f}(eg) = \tilde{f}(e)g$ for $e \in E$, $\gamma \in \Gamma$, $g \in G$.

Denote it by $T\Gamma EG_{(\theta, c)}(X)$

$\mathrm{Int}_s : \Gamma \to \mathrm{Int}(G)$
$\gamma \mapsto \mathrm{Int}_{s(\gamma)}$

Prop: let $s : \Gamma \to G$ a map making $\mathrm{Int}_s\, \theta : \Gamma \to \mathrm{Aut}\, G$ a group homomorphism.

let $C_s : \Gamma \times \Gamma \to Z$ the 2 cocycle defined last talk

$$(\gamma, \gamma') \mapsto s(\gamma)\theta_\gamma(s(\gamma'))s(\gamma\gamma')^{-1}$$

Then $T\Gamma EG_{(\theta,c)}(X)$ is equivalent to $T\Gamma EG_{(\mathrm{Int}_s\theta,\, cC_s)}(X)$.

"pf": follows from the equivalence btw the categories $\mathcal{C}(\theta, c)$ and $\mathcal{C}(\mathrm{Int}_s\, \theta, cC_s)$ (proved last talk) where

$\mathcal{C}(\theta, c)$ is the cat. of pairs $(M, \cdot)$ consisting of a set M and a (θ, c)-twisted (G, Γ)-action on M whose morphisms are (G, Γ)-equiv. maps. ▨.

Def: let $\pi : E \to Y$ a (θ, c)-twisted Γ equiv. G-bundle

let $f : X \to Y$ a Γ-equiv. map.

. $f^*E := \{(x, e) \in X \times E \mid f(x) = \pi(e)\}$

. Twisted Γ-action: $(x, e) \cdot \gamma \overset{(\star_1)}{=} (x \cdot \gamma, e \cdot \gamma)$ (given by coordinate-wise action)

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(this gives the structure of principal G-bundle to f^*E)

can check $\rightsquigarrow$
- $f^*E \to X$ is a (θ, c)-twisted Γ-equiv. bundle
- $f^*E \to E$ covering $X \to Y$ is a morphism of (θ, c)-twisted Γ-equiv principal G-bundles.

Let $E \to X$ a principal G-bundle.

Let M a set with $G \circlearrowleft M$

[if left action $G \circlearrowleft M$, convert it to right action $m.g := g^{-1}.m$. This works b/c $(m.g.)g' = m.(g.g')$.

$$g^{-1}.(g'^{-1}.m) = (g^{-1}g'^{-1}).m$$
$$\parallel$$
$$(m.g')g = m.g'g$$]

- $E(M) := (E \times M)/G$ (the orbit space)

- If $G \circlearrowleft M$ is a left action, then $E(M) = E \times_G M$

with $(eg, m) \sim (e, g.m)$ for $e \in E, g \in G, m \in M$

<u>Prop:</u> Let M be a (θ, c)-twisted right (G, Γ)-manif.

" $\pi : E \to X$ be a (θ, c) twisted Γ-equiv. principal G-bundle.

View E as the G-frame bundle of $E(M)$ via:

$$\begin{aligned} M &\xrightarrow{\cong} E(M)_x \\ m &\mapsto [e, m] \end{aligned}$$

Then the associated fiber bdle with typical fiber M

$$\begin{aligned} \alpha : E(M) := E \times_G M &\longrightarrow X \\ [e, m] &\longmapsto \pi(e) \end{aligned}$$

given by the above construction is a Γ-equiv. fiber bdle.

<u>proof:</u> we have $[e, m].\gamma = [e.\gamma, m.\gamma]$

(this follows from $\{\hat{G}$ actions on a set $M\} \overset{bij}{\Longleftrightarrow} \left\{\begin{matrix}(\theta, c)\text{-twisted} \\ (G, \Gamma)\text{-action} \\ \text{on } M\end{matrix}\right\}$ last time

$\rightsquigarrow$ Γ- action on $E(M)$.

Moreover, $\alpha([e,m].\gamma) = \alpha([e.\gamma, m.\gamma]) = \pi(e.\gamma) = \pi(e).\gamma$

(last equality by Γ- equiv. of π, by def. of a (θ, c) twisted Γ- equiv principal G- bdle). $\boxtimes$

<u>Def:</u> Let $s : X \longrightarrow F$ be a smooth section of a twisted Γ- equiv. fibre bundle $F \longrightarrow X$,

- s is said twisted Γ- equiv. if $s(x.\gamma) = s(x).\gamma$, $x \in X, \gamma \in \Gamma$
with obvious modification in the case of left actions.

$C^\infty(X, F)^\Gamma$:= space of twisted Γ- equiv. smooth sections of a twisted Γ- equiv. fibre bundle $F \longrightarrow X$.

- $C^\infty(E, M)^{G,\Gamma}$:= Space of (θ, c)-twisted (G,Γ)- equiv. smooth maps $\tilde{s} : E \longrightarrow M$.
(i.e., s.t. $f(eg) = f(m).g$, $f(e.\gamma) = f(e).\gamma$)

<u>prop:</u> $C^\infty(E, M)^{G,\Gamma} \xrightarrow{\text{bij}} C^\infty(X, E(M))^\Gamma$

<u>proof:</u>

$$\{\text{smooth sections } s : X \to E(M)\} \underset{\sim}{\xrightarrow{\Psi}} \left\{ \begin{array}{l} \text{smooth maps } \tilde{s} : E \longrightarrow M \text{ s.t.} \\ \tilde{s}(eg) = \tilde{s}(e).g, \ \forall e \in E, \forall g \in G \ (\star) \end{array} \right\}$$

Let $\tilde{s} : E \longrightarrow M$ satisfying $(\star)$

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Then $(Id, \tilde{s}) : E \to E \times M$ satisfies

$$(Id, \tilde{s})(eg) = (eg, \tilde{s}(eg)) = (eg, \tilde{s}(e).g)$$

$$\hspace{6em} \parallel$$

$$(Id, \tilde{s})(e).g = (e, \tilde{s}(e)).g \underset{\star_1}{=} (eg, \tilde{s}(e)).g$$

(coordinate-wise action p3)

Hence, $(Id, \tilde{s}) : E \to E \times M$ is G-equiv. hence descends to the quotient $E(M)$.

$\rightsquigarrow$

$$\begin{tikzcd}
E \arrow[r, "\tilde{s}"] \arrow[d] \arrow[dr, phantom, "\bigcirc"] & E \times M \arrow[d] \\
X = E/G \arrow[r, "s"'] & E(M)
\end{tikzcd}$$

$\rightsquigarrow$ $s :=$ the section corresponding to $\tilde{s}$.

Conversely take $s : E/G \longrightarrow E(M)$ where $s([e]) = [(e, m)]$

set $\tilde{s}(e) := m$ (well defined since fibers of $E \to X$ are G-torsors)

for Γ-equivariance, use. previous talk's result:

$\tilde{s}$ is $\hat{G}$-equiv $\Longleftrightarrow$ s is Γ-equiv.

$\blacksquare$

Prop: Let $H \subseteq G$ a Lie subgrp which is preserved by the Γ-action.

(i.e. $\gamma . h \in H$, $\forall h \in H$)

consider the usual $G \circlearrowright G/H$ (via $g_1 . (gH) = (g_1 g) H$).

Then $\gamma . (gH) = (\gamma . g) H$ defines an action of $G \rtimes_\theta \Gamma$

on G/H defined by θ.

proof: $\forall g, g' \in G$, $\forall \gamma, \gamma' \in \Gamma$ we have:

$$\gamma.(g'gH) = \gamma.(g'g)H = \theta_\gamma(g'g)H = \theta_\gamma(g')\theta_\gamma(g)H$$

↑ Γ acts on G via θ

$$= \theta_\gamma(g')(\gamma.gH)$$

and $\gamma.(\gamma'.gH) = \theta_\gamma\theta_{\gamma'}(g)H = \theta_{\gamma\gamma'}(g)H = \gamma\gamma'.gH$. $\square$

Let $H \subset G$ a Γ-invariant subgrp.

Let σ a section of $E(G/H)$ (a "reduction of structure group")

$$\begin{tikzcd} E \arrow[r] & E/H \\ \sigma^*E \arrow[u] \arrow[r] & X \arrow[u, "\sigma"'] \end{tikzcd}$$

View E as the bundle of G-frames of itself.

$E \times G/H \cong E/H$ & $E \to E/H$ is a principal H-bundle.

Then $E_\sigma := \sigma^*E$ is a principal H-bundle over X

<u>Facts</u> • E_σ is Γ invariant $\iff$ σ is Γ-invariant.

• σ is Γ-inv. $\implies$ $\exists$ induced (θ, c)-twisted Γ-equiv. structure on E_σ.

This motivates the following def:

<u>Def:</u> Given $E \in T\Gamma EG_{(\theta,c)}(Y)$, and $H \subset G$ Γ-invariant, a (θ,c)-twisted Γ-equiv. reduction of structure group of E to H is a Γ-invariant section of $E(G/H)$.